Double Categories Exercises
ACT 2023 Tutorial

Dorette Pronk

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Double Categorical Concepts

1. Write out the conditions that the unitor and associator cells for proarrow composition need to satisfy.

2. Show that transformations $\beta\colon F \Rightarrow G\colon \mathbb{C} \rightrightarrows \mathbb{D}$ with arrow components can also be viewed as compatible pairs of transformations

$$\beta_0\colon F_0 \Rightarrow G_0 \qquad \beta_1\colon F_1 \Rightarrow G_1$$

   Write out the compatibility requirements.

3. Given that $F, G, H, K\colon \mathbb{C} \to \mathbb{D}$ are pseudo double functors, $\alpha\colon F \Rightarrow H$ and $\beta\colon G \Rightarrow K$ are arrow-valued transformations, and $\zeta\colon F \Rightarrow G$ and $\theta\colon H \Rightarrow K$ are proarrow-valued transformation, What are the requirements we need on a family of cells Ω_X,

$$\begin{array}{ccc} FX & \overset{\zeta_X}{-\!\!\!\mid\!\!\!\to} & GX \\ {\scriptstyle \alpha_X}\downarrow & \Omega_X & \downarrow{\scriptstyle \beta_X} \\ HX & \underset{\theta_X}{-\!\!\!\mid\!\!\!\to} & KX \end{array}$$

   to form a modification?

Examples

1. There is a way to make a double category of spans for a 2-category with comma squares, where the double cells are of the form

$$\begin{array}{ccccc} X & \overset{\ell}{\longleftarrow} & S & \overset{r}{\longrightarrow} & Y \\ {\scriptstyle u}\downarrow & \alpha & {\scriptstyle m}\downarrow & \beta & \downarrow{\scriptstyle v} \\ X' & \underset{\ell'}{\longleftarrow} & S' & \underset{r'}{\longrightarrow} & Y' \end{array}$$

   Define the two types of composition for this double category and check the compatibility axioms. Note that this is not the double 2-category of spans that we will introduce later in this tutorial.

2. A double category is called *thin* or *flat* if any compatible square of arrows and proarrows can be filled with at most one double cell. Check that the double category of relations is thin. Can you construct other thin double categories.

3. Convince yourself that a double category where the object category is the terminal category can indeed be viewed as a monoidal category.

Limits in Double Categories (Based on the paper *Limits in Double Categories* by M. Grandis and R. Paré.

1. Double comma squares are comma squares in the 2-category of double categories once we have chosen a direction for the double cells we will use. We will use here the arrow-valued ones. Let $F\colon \mathbb{A} \longrightarrow \mathbb{X}$ be a (pseudo) double functor and $X\colon \mathbb{T} \longrightarrow \mathbb{X}$ be a double functor from the terminal double category $\mathbb{T}$ with one object, on identity arrow, one identity proarrow and one identity double cell, picking out an object X in $\mathbb{X}$. Write out in detail what the comma double category $(F \downdownarrows X)$ is:

$$\begin{array}{ccc} (F \downdownarrows X) & \xrightarrow{F'} & \mathbb{T} \\ {\scriptstyle \Pi}\downarrow & \overset{\varphi}{\Rightarrow} & \downarrow{\scriptstyle X} \\ \mathbb{A} & \xrightarrow[F]{} & \mathbb{X} \end{array}$$

2. The (arrow) double terminal object of a double category $\mathbb{A}$ is an object T with for each object A a unique arrow $!_A\colon A \to T$ and for each proarrow $A \overset{f}{\longrightarrow\!\!\!\!\!\!\!+} B$ a unique cell

$$\begin{array}{ccc} A & \overset{f}{\longrightarrow\!\!\!\!\!\!\!+} & B \\ {\scriptstyle !_A}\downarrow & {\scriptstyle !_f} & \downarrow{\scriptstyle !_B} \\ T & = \!= & T \end{array}$$

Identify examples of double categories with and without a terminal object.

3. Given an indexing diagram $\mathbb{I}$, consider the hom-double category (the cotensor) $\mathbb{A}^{\mathbb{I}}$ and the diagonal double functor $D\colon \mathbb{A} \to \mathbb{A}^{\mathbb{I}}$, taking each object to the constant double functor with that object as its value. Now an arrow-valued double cone for $F\colon \mathbb{I} \to \mathbb{A}$ is a pair of an object A in $\mathbb{A}$ and an arrow-valued transformation $\alpha\colon DA \Rightarrow F$. Write out the data and conditions to describe such a cone.

4. Universality of such a cone is equivalent to requiring that it be the terminal object of $(D \downdownarrows f)$. Write out what this means. (Note that this does not imply functoriality in the proarrow direction!)

5. The tabulator of a proarrow is its arrow-valued limit and its cotabulator is its arrow valued colimit. Calculate examples of these, for instance in the double category of sets and relations and in the double category of categories and profunctors.

6. Show that a 2-category is 2-complete iff its double category of quintets has all double limits. If one takes the horizontal or vertical double category for a 2-category, the double limits also give the weighted limits.

Properties of Fibrations

1. Extend the Fam-construction to a double category. Is this again a fibration? What other properties does it have?