Intro to Double Categories

leading to double fibrations*

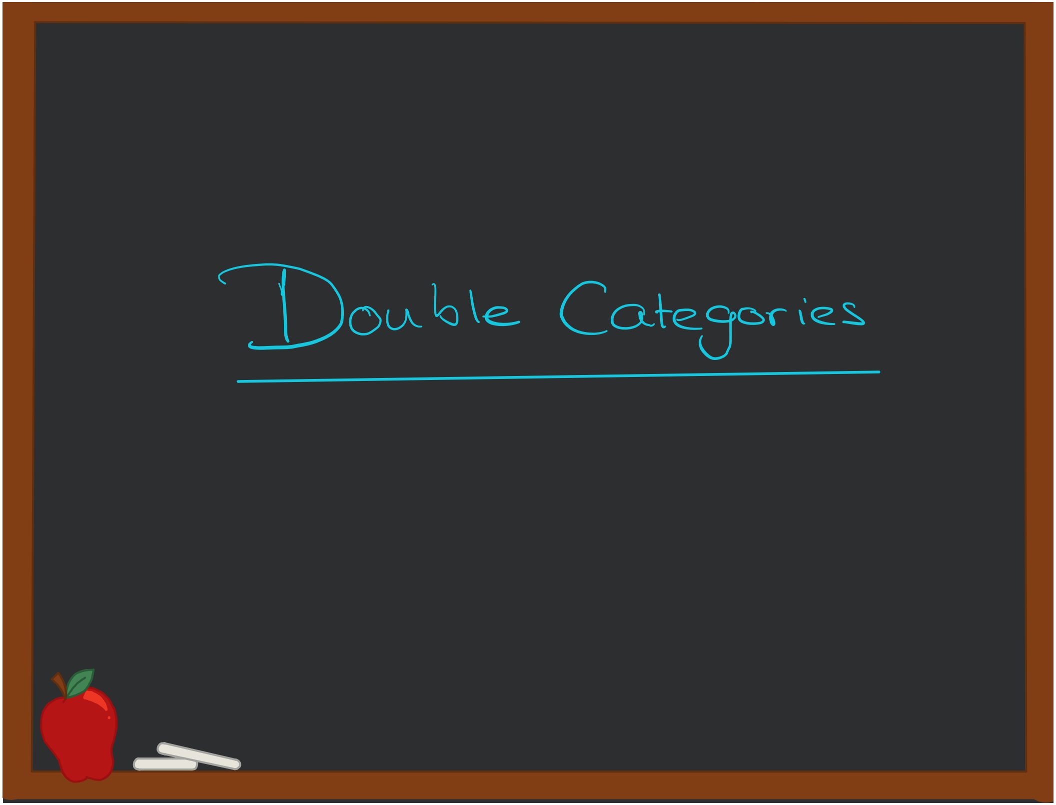
Dorette Pronk (Dalhousie University)

Tutorial at ACT 2023

based on:

- papers by Marco Grandis and Robert Paré
- Double Fibrations - joint work with Geoff Cruttwell, Michael Lambert, and Martin Szyld

* Feel free to ask any questions you have about double categories!


Double Categories

Intuitively:

A double category has:

- objects
- two types of arrows, each with its own composition structure, say $\xrightarrow{\;\shortmid\;}$ and $\longrightarrow$
- double cells:

$$\begin{tikzcd} {} \arrow[r, "\shortmid" marking] \arrow[d] \arrow[dr, phantom, "\alpha"] & {} \arrow[d] \\ {} \arrow[r, "\shortmid" marking] & {} \end{tikzcd}$$

that can be composed in both directions and these compositions need to be compatible.

Definition For a 2-category $\mathcal{K}$, a **pseudo category** $\mathbb{C}$ in $\mathcal{K}$ consists of a diagram:

$$
\begin{tikzcd}
C_1 \times_{C_0} C_1 \arrow[r, "\otimes"] & C_1 \arrow[r, shift left=2, "s"] \arrow[r, shift right=2, "t"'] & C_0 \arrow[l, "y" description]
\end{tikzcd}
$$

and iso. 2-cells:

$$
\begin{tikzcd}
C_1 \times_{C_0} C_1 \times_{C_0} C_1 \arrow[r, "1\times\otimes"] \arrow[d, "\otimes\times 1"'] \arrow[dr, phantom, "\overset{\alpha}{\cong}"] & C_1 \times_{C_0} C_1 \arrow[d, "\otimes"] \\
C_1 \times_{C_0} C_1 \arrow[r, "\otimes"'] & C_1
\end{tikzcd}
\qquad
\begin{tikzcd}
C_1 \arrow[r, "{(y,1)}"] \arrow[dr, bend right, "1_{C_1}"'] \arrow[dr, phantom, shift right=3, "\overset{l}{\cong}"] & C_1 \times_{C_0} C_1 \arrow[d, "\otimes"] & C_1 \arrow[l, "{(1,y)}"'] \arrow[dl, bend left, "1_{C_1}"] \arrow[dl, phantom, shift left=3, "\overset{r}{\cong}"] \\
& C_1 &
\end{tikzcd}
$$

normalized: l and r ids.

Double Categories

Definition A double category is a pseudo category object in Cat :

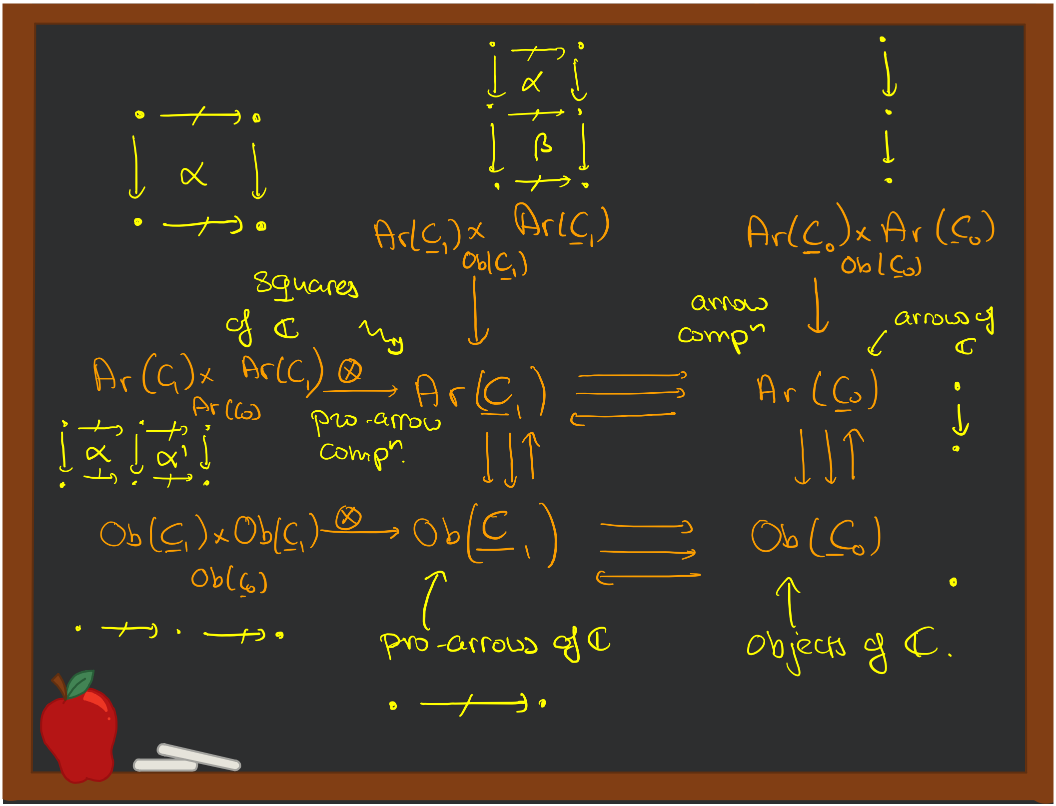
$$\underline{C}_1 \times_{\underline{C}_0} \underline{C}_1 \xrightarrow{\otimes} \underline{C}_1 \underset{y}{\overset{s}{\underset{t}{\rightrightarrows}}\atop\longleftarrow} \underline{C}_0$$

$\otimes$ ps. assoc. and ps. unitary

$\underline{C}_0$, $\underline{C}_1$ categories.

Let's spell that out :

$$
\begin{tikzcd}
 & Ar(C)_1 \times_{Ob(C)_1} Ar(C)_1 \arrow[d] & & Ar(\underline{C}_0) \times_{Ob(\underline{C}_0)} Ar(\underline{C}_0) \arrow[d] \\
Ar(\underline{C}_1) \times_{Ar(\underline{C}_0)} Ar(\underline{C}_1) \arrow[r, "\otimes"] & Ar(\underline{C}_1) \arrow[r, shift left=2] \arrow[r] \arrow[d, shift right=2] \arrow[d] & Ar(\underline{C}_0) \arrow[l, shift left=2] \arrow[d, shift right=2] \arrow[d] & Ar(\underline{C}_0) \arrow[d, shift right=2] \arrow[d] \\
Ob(\underline{C}_1) \times_{Ob(\underline{C}_0)} Ob(\underline{C}_1) \arrow[r, "\otimes"] & Ob(\underline{C}_1) \arrow[r, shift left=2] \arrow[r] \arrow[u, shift right=2] & Ob(\underline{C}_0) \arrow[l, shift left=2] \arrow[u, shift right=2] & Ob(\underline{C}_0) \arrow[u, shift right=2]
\end{tikzcd}
$$


α
α
β
Ar(C1) x Ob(C1) Ar(C1)
Ar(C0) x Ob(C0) Ar(C0)
squares of C
arrow compn
arrows of C
Ar(C1) x Ar(C0) Ar(C1)
⊗
pro-arrow compn.
Ar(C1)
Ar(C0)
α
α'
Ob(C1) x Ob(C0) Ob(C1)
⊗
Ob(C1)
Ob(C0)
pro-arrows of C
Objects of C.

Proarrow composition is only required to be unitary and associative up to isomorphism:

these are double cells:

unitors:

$$\begin{tikzcd} A \arrow[r, "y_B h", "|" marking] \arrow[d, "1_A"'] \arrow[dr, phantom, "\lambda_h"] & B \arrow[d, "1_B"] \\ A \arrow[r, "h"', "|" marking] & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "h y_A", "|" marking] \arrow[d, "1_A"'] \arrow[dr, phantom, "\rho_h"] & B \arrow[d, "1_B"] \\ A \arrow[r, "h"', "|" marking] & B \end{tikzcd}$$

associators:

$$\begin{tikzcd} A \arrow[r, "gf", "|" marking] \arrow[d, "1_A"'] \arrow[drr, phantom, "\alpha_{hgf}"] & C \arrow[r, "h", "|" marking] & D \arrow[d, "1_D"] \\ A \arrow[r, "f"', "|" marking] & B \arrow[r, "hg"', "|" marking] & D \end{tikzcd}$$

- functorial w.r.t. dbl cells
- Vertically invertible
- satisfying coherence.

Examples

1. **Rel** : objects are sets

proarrows are relations

arrows are functions

squares :

$$\begin{tikzcd} S \arrow[r, "R", "/"{marking}] \arrow[d, "u"'] \arrow[dr, phantom, "\Downarrow"] & T \arrow[d, "v"] \\ S' \arrow[r, "R'"', "/"{marking}] & T' \end{tikzcd}$$

$R \subseteq S \times T$

$R' \subseteq S' \times T'$

such that

for $(s,t) \in R$, $\big(u(s), v(t)\big) \in R'$

2. $\mathrm{Span}(\underline{\mathcal{C}})$, where $\underline{\mathcal{C}}$ is a category with pullbacks:

<u>obj</u>: are objects in $\mathcal{C}$

<u>arrows</u> are arrows in $\underline{\mathcal{C}}$.

<u>Proarrows</u> are spans

$$\begin{tikzcd} S & A \arrow[l, "s"'] \arrow[r, "t"] & T \end{tikzcd}$$

<u>squares</u>: commutative diagrams

$$\begin{tikzcd} S \arrow[d, "u"'] & A \arrow[l, "s"'] \arrow[r, "t"] \arrow[d, "w"] & T \arrow[d, "v"] \\ S' & A' \arrow[l, "s'"'] \arrow[r, "t'"] & T' \end{tikzcd}$$

<u>3.</u> For $\underline{C}_0 = \underline{1}$, a double category $\underline{C}_1 \times \underline{C}_1 \xrightarrow{\otimes} \underline{C}_1 \xleftarrow{y} \underline{1}$ is just a <u>monoidal category</u>.

4. A 2-category $\mathcal{A}$ gives rise to double categories

- $\mathbb{V}\mathcal{A}$ with cells
$$\begin{tikzcd} A \arrow[r, "1"] \arrow[d, "u"'] \arrow[dr, phantom, "\alpha"] & A \arrow[d, "v"] \\ B \arrow[r, "1"'] & B \end{tikzcd}$$
when
$$\begin{tikzcd} A \arrow[d, bend right, "u"'] \arrow[d, bend left, "v"] \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] \\ B \end{tikzcd}$$
in $\mathcal{A}$.

- $\mathbb{H}\mathcal{A}$ with cells
$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "1"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d] \\ A \arrow[r, "k"'] & B \end{tikzcd}$$
when
$$\begin{tikzcd} A \arrow[r, bend left, "h"] \arrow[r, bend right, "k"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}$$
in $\mathcal{A}$.

- $\mathbb{Q}\mathcal{A}$ with cells
$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "u"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "v"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$
when
$$\begin{tikzcd} A \arrow[r, bend left, "vh"] \arrow[r, bend right, "ku"'] \arrow[r, phantom, "\Downarrow\alpha"] & D \end{tikzcd}$$
in $\mathcal{A}$.

Note: if $\mathcal{H}$ and $\mathcal{V}$ are wide subcategories of $\mathcal{A}$, there is an obvious subdouble category of quintets with horizontal arrows in $\mathcal{H}$ and vertical arrows in $\mathcal{V}$.

Pseudo Double Functors

For: double categories $\mathbb{C}$: $\underline{C}_1 \times_{\underline{C}_0} \underline{C}_1 \xrightarrow{\otimes} \underline{C}_1 \underset{\xleftarrow{\ y\ }}{\overset{s}{\underset{t}{\rightrightarrows}}} \underline{C}_0$

and $\mathbb{D}$: $\underline{D}_1 \times_{\underline{D}_0} \underline{D}_1 \xrightarrow{\otimes} \underline{D}_1 \underset{\xleftarrow{\ y\ }}{\overset{s}{\underset{t}{\rightrightarrows}}} \underline{D}_0$

Strict in the arrow direction, pseudo in the proarrow direction

a pseudo double functor consists of

$F_0 : \underline{C}_0 \longrightarrow \underline{D}_0$ and $F_1 : \underline{C}_1 \longrightarrow \underline{D}_1$

with comparison cells:

$$
\begin{tikzcd}
\underline{C}_1 \times_{\underline{C}_0} \underline{C}_1 \arrow[r, "\otimes"] \arrow[d, "F_1 \times_{F_0} F_1"'] \arrow[dr, phantom, "\overset{\varphi_1}{\Rightarrow}\ \cong"] & \underline{C}_1 \arrow[d, "F_1"] \\
\underline{D}_1 \times_{\underline{D}_0} \underline{D}_1 \arrow[r, "\otimes"'] & \underline{D}_1
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{C}_0 \arrow[r, "y"] \arrow[d, "F_0"'] \arrow[dr, phantom, "\overset{2}{\Rightarrow}\ \cong"] & \underline{C}_1 \arrow[d, "F_1"] \\
\underline{D}_0 \arrow[r, "y"'] & \underline{D}_1
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{C}_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] \arrow[d, "F_1"'] \arrow[dr, phantom, "="] & \underline{C}_0 \arrow[d, "F_0"] \\
\underline{D}_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & \underline{D}_0
\end{tikzcd}
$$

Stricter!

Subject to the usual coherence condns

Transformations between double functors

These come in two flavours:

- with proarrow components — these are the internal transformations between internal functors:

$$\begin{tikzcd}
C_1 \arrow[r, shift left, "G_1"] \arrow[r, shift right, "F_1"'] \arrow[d, shift right=3, "s"'] \arrow[d, shift left=3, "t"] & D_1 \arrow[d, shift right=3] \arrow[d, shift left=3] \\
C_0 \arrow[u, "y"] \arrow[ur, "\alpha"] \arrow[r, shift left, "G_0"] \arrow[r, shift right, "F_0"'] & D_0 \arrow[u]
\end{tikzcd}$$

$\alpha : F \nRightarrow G$

$\alpha : \mathcal{C}_0 \longrightarrow D_1$ is a functor

- for each object X in $\mathbb{C}$,

  $\alpha_X : F_0 X \longrightarrow G_0 X$ is a proarrow in $\mathbb{D}$

- for each arrow $v : X \longrightarrow Y$ in $\mathbb{C}$, there is a double cell

$$\begin{tikzcd}
F_0 X \arrow[r, "\alpha_X", "|" marking] \arrow[d, "F_0 v"'] \arrow[dr, phantom, "\alpha_v"] & G_0 X \arrow[d, "G_0 v"] \\
F_0 Y \arrow[r, "\alpha_Y", "|" marking] & G_0 Y
\end{tikzcd}$$

these need to satisfy:

- horizontal / proarrow naturality

$$
\begin{tikzcd}
F_0 X \arrow[r, "F_1 h", "|" marking] \arrow[d, "F_0 v"'] \arrow[dr, phantom, "F_1 \theta"] & F_0 X' \arrow[r, "\alpha_{X'}", "|" marking] \arrow[d, "F_0 v'"] \arrow[dr, phantom, "\alpha_{v'}"] & G_0 X' \arrow[d, "G_0 v'"] \\
F_0 Y \arrow[r, "F_1 k"', "|" marking] & F_0 Y' \arrow[r, "\alpha_{Y'}"', "|" marking] & G_0 Y'
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
F_0 X \arrow[r, "\alpha_X", "|" marking] \arrow[d, "F_0 v"'] \arrow[dr, phantom, "\alpha_v"] & G_0 X \arrow[r, "G_1 h", "|" marking] \arrow[d, "G_0 v"] \arrow[dr, phantom, "G_1 \theta"] & G_0 X' \arrow[d, "G_0 v'"] \\
F_0 Y \arrow[r, "\alpha_Y"', "|" marking] & G_0 Y \arrow[r, "G_1 k"', "|" marking] & G_0 Y'
\end{tikzcd}
$$

- vertical / arrow functoriality: $\alpha_{wv} = \alpha_w \alpha_v$

- The transformations with arrow components are defined dually, with components

$$\begin{tikzcd}
F_0 X \arrow[r, "F_1 h", "\shortmid"{marking}] \arrow[d, "\beta_X"'] \arrow[dr, phantom, "\beta_h"] & F_0 X' \arrow[d, "\beta_{X'}"] \\
G_0 X \arrow[r, "G_1 h"', "\shortmid"{marking}] & G_0 X'
\end{tikzcd}$$

that need to be natural in the arrow direction:

$$\begin{tikzcd}
F_0 X \arrow[r, "F_1 h", "\shortmid"{marking}] \arrow[d, "\beta_X"'] \arrow[dr, phantom, "\beta_h"] & F_0 X' \arrow[d, "\beta_{X'}"] \\
G_0 X \arrow[r, "G_1 h"', "\shortmid"{marking}] \arrow[d, "G_0 v"'] \arrow[dr, phantom, "G_1 \theta"] & G_0 X' \arrow[d, "G_0 v'"] \\
G_0 Y \arrow[r, "G_1 k", "\shortmid"{marking}] & G_0 Y'
\end{tikzcd}
=
\begin{tikzcd}
F_0 X \arrow[r, "F_1 h", "\shortmid"{marking}] \arrow[d, "F_0 v"'] \arrow[dr, phantom, "F_1 \theta"] & F_0 X' \arrow[d, "F_0 v'"] \\
F_0 Y \arrow[r, "F_1 k", "\shortmid"{marking}] \arrow[d, "\beta_Y"'] \arrow[dr, phantom, "\beta_k"] & F_0 Y' \arrow[d, "\beta_{Y'}"] \\
G_0 Y \arrow[r, "G_1 k", "\shortmid"{marking}] & G_0 Y'
\end{tikzcd}$$

and functorial in the proarrow direction:

$$
\begin{tikzcd}
F_0 X \arrow[rr, "F_1(g \cdot h)", "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\varphi_{g,h}"] & & F_0 Z \arrow[d, equal] \\
F_0 X \arrow[r, "F_1 h", "\shortmid"{marking}] \arrow[d, "\beta_X"'] \arrow[dr, phantom, "\beta_h"] & F_0 Y \arrow[r, "F_1 g", "\shortmid"{marking}] \arrow[d, "\beta_Y"] \arrow[dr, phantom, "\beta_g"] & F_0 Z \arrow[d, "\beta_Z"] \\
G_0 X \arrow[r, "G_1 h"', "\shortmid"{marking}] & G_0 Y \arrow[r, "G_1 g"', "\shortmid"{marking}] & G_0 Z
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
F_0 X \arrow[rr, "F_1(gh)", "\shortmid"{marking}] \arrow[d, "\beta_X"'] \arrow[drr, phantom, "\beta_{gh}"] & & F_0 Z \arrow[d, "\beta_Z"] \\
G_0 X \arrow[rr, "G_1(gh)"', "\shortmid"{marking}] \arrow[d, equal] \arrow[drr, phantom, "\psi_{gh}"] & & G_0 Z \arrow[d, equal] \\
G_0 X \arrow[r, "G_1 h"', "\shortmid"{marking}] & G_0 Y \arrow[r, "G_1 g"', "\shortmid"{marking}] & G_0 Z
\end{tikzcd}
$$

The category of double categories

- objects: double categories $\mathbb{A}, \mathbb{B}, \mathbb{C}, \ldots$
- arrows: pseudo double functors
- transformations:
  - proarrow valued
  - arrow valued
- modifications

Modifications

Given $F, G, H, K : \mathbb{C} \longrightarrow \mathbb{D}$

with

$$\begin{tikzcd} F \arrow[r, Rightarrow, "\alpha", "/" marking] \arrow[d, Rightarrow, "\beta"'] \arrow[dr, phantom, "\Omega"] & G \arrow[d, Rightarrow, "\gamma"] \\ H \arrow[r, Rightarrow, "\delta"', "/" marking] & K \end{tikzcd}$$

α, δ proarrow transformations

β, γ arrow transformations

a modification Ω is given by

a double cell

$$\begin{tikzcd} FX \arrow[r, "\alpha_X", "/" marking] \arrow[d, "\beta_X"'] \arrow[dr, phantom, "\Omega_X"] & GX \arrow[d, "\gamma_X"] \\ HX \arrow[r, "\delta_X"', "/" marking] & KX \end{tikzcd}$$

in $\mathbb{D}$ for each object X in $\mathbb{C}$.

natural in both directions

Results:

- $\mathrm{Hom}(\mathbb{C}, \mathbb{D})$ has the structure of a double category.
- By restricting to globular cells we can make this a category.
- DblCat is naturally enriched over itself; we can also make it a 2-category.
- There are ways to make it a double category.

The Grothendieck Construction

(category of elements)

- many cool properties (see n-lab, listen to talks from this year's CT conference)
- today: I will mostly focus on two of them

Recall a classical result from SGA 4:

- For an indexing pseudo functor $\boxed{F : \mathcal{A}^{op} \longrightarrow \underline{Cat}}$ with structure isomorphisms

$$\varphi_A : 1_{FA} \overset{\sim}{\Rightarrow} F(1_A)$$

$$\varphi_{g,f} : Fg \circ Ff \overset{\sim}{\Rightarrow} F(g \circ f)$$

the category of elements $\boxed{\int_{\mathcal{A}} F} \xrightarrow{\pi_F} \mathcal{A}$ has

objects: (A, x), $A \in Obj(\mathcal{A})$, $x \in Obj(F(A))$

arrows: $(g, \psi) : (A, x) \longrightarrow (B, y)$ with $g : A \to B$ and $\psi : x \longrightarrow F(g)(y)$ in $F(A)$.

identities : $id_{(A,x)} = (id_A, (\varphi_A)_x)$

$(\varphi_A)_x : x \to F(1_A)(x)$

composition:

for $(A,x) \xrightarrow{(g_1,\psi_1)} (B,y) \xrightarrow{(g_2,\psi_2)} (C,z)$:

$$(g_2,\psi_2) \circ (g_1,\psi_1) = (g_2 \circ g_1, \varphi_{g_1 g_2} \circ F(g_1)(\psi_2) \circ \psi_1)$$

$$x \xrightarrow{\psi_1} F(g_1)(y) \xrightarrow{F(g_1)(\psi_2)} F(g_1)F(g_2)(z) \xrightarrow{\varphi_{g_1,g_2}} F(g_1 g_2)(z)$$

Properties of $\int_{\mathcal{A}} F \xrightarrow{\pi_F} \mathcal{A}$.

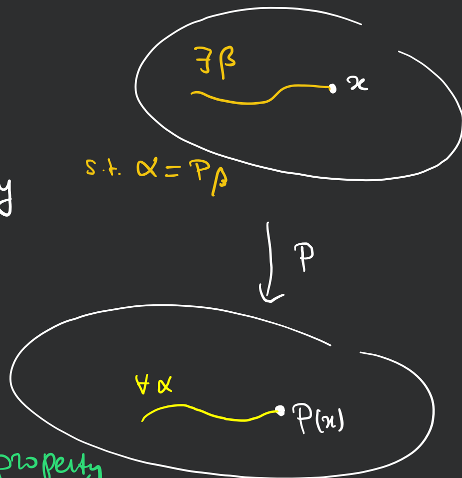
- $\pi_F : \int_{\mathcal{A}} F \longrightarrow \mathcal{A}$ is defined by:

$$
\begin{aligned}
(A, x) &\longmapsto A \\
(g, \varphi) &\longmapsto g
\end{aligned}
$$

- π_F is a <u>fibration</u>

Fibrations

basic idea from topology



β is a lifting of α

path lifting property

Fibrations – Cartesian Arrows

Let $p: \underline{E} \longrightarrow \underline{B}$ be a functor. An arrow $\gamma: e' \to e$ in $\underline{E}$ is <u>Cartesian</u> if for all

$$
\begin{tikzcd}
e'' \arrow[d, dashed, "\exists!"', "\chi"'] \arrow[dr, "\psi"] \arrow[dr, phantom, "=" near start, shift right=3ex] & \\
e' \arrow[r, "\gamma"'] & e
\end{tikzcd}
\qquad \text{in } \underline{E}
$$

$$
\begin{tikzcd}
pe'' \arrow[d, "p(\chi) = g"'] \arrow[dr, "p\psi"] \arrow[dr, phantom, "=" near start, shift right=3ex] & \\
pe' \arrow[r, "p\gamma"'] & pe
\end{tikzcd}
\qquad \text{in } \underline{B}
$$

there is a unique lifting χ.

Fibrations

① $p : \underline{E} \longrightarrow \underline{B}$ is a <u>fibration</u> if for any

$$e' \xrightarrow{\varphi} e \qquad \text{in } \underline{E}$$

$$b \xrightarrow[f]{} pe \qquad \text{in } \underline{B}$$

there is a Cartesian lifting φ (of f to e)

② A <u>cleavage</u> for p consists of a choice of a Cartesian lifting for each f and e.

π_F is a fibration

$F : \mathcal{A}^{op} \longrightarrow \underline{Cat}$

$$\begin{tikzcd}
\int_{\mathcal{A}} F \arrow[d, "\pi_F"'] & (B, \quad) \arrow[r] & (A, x) \\
\mathcal{A} & B \arrow[r, "f"'] & A
\end{tikzcd}$$

$x \in FA$

$\in FB$

π_F is a fibration

$$\begin{tikzcd} \int_{\mathcal{A}} F \arrow[d, "\pi_F"'] & (B, Ff(x)) \arrow[r, "{(f,\, 1_{Ff(x)})}"'] & (A, x) \\ \mathcal{A} & B \arrow[r, "f"'] & A \end{tikzcd}$$

Canonical Cartesian morphisms:

$$(f,\ 1_{Ff(x)})$$

they form a cleavage.

Result: for a fixed base category $\underline{B}$, the Grothendieck construction extends to a 2-equivalence:

$$\int_B : [\underline{B}^{op}, \underline{Cat}] \xrightarrow{\sim} \underline{Fib}(\underline{B})$$

$[\underline{B}^{op}, \underline{Cat}]$: pseudo-functors, pseudo natural transformations, modifications

$\underline{Fib}(\underline{B})$: fibrations over $\underline{B}$, cartesian functors, vertical natural transformations

$$\begin{tikzcd} \underline{E} \arrow[rr, bend left, "\Phi"] \arrow[rr, bend right, "\Psi"'] \arrow[rr, phantom, "\Downarrow\alpha"] \arrow[dr, "p"'] & & \underline{F} \arrow[dl, "q"] \\ & \underline{B} & \end{tikzcd} \qquad q(\alpha_e) = 1_{p(e)}.$$

Fibrations over an arbitrary base

$\underline{\mathrm{Fib}}$ is the 2-category:

<u>objects</u>: cloven fibrations $P: \underline{E} \to \underline{B}$ (arbitrary $\underline{B}$)

<u>arrows</u>: $f : (P : \underline{E} \to \underline{B}) \longrightarrow (P' : \underline{E}' \to \underline{B}')$

is a commutative square:

$$\begin{tikzcd}
\underline{E} \arrow[r, "f^{\top}"] \arrow[d, "P"'] & \underline{E}' \arrow[d, "P'"] \\
\underline{B} \arrow[r, "f^{\perp}"'] & \underline{B}'
\end{tikzcd}$$

, $f^{\top}$ preserves Cartesian arrows.

<u>2-cells</u>: $\alpha : f \Rightarrow g$:

$$\begin{tikzcd}
\underline{E} \arrow[r, bend left, "f^{\top}"] \arrow[r, bend right, "g^{\top}"'] \arrow[r, phantom, "\Downarrow \alpha^{\top}"] \arrow[d, "P"'] & \underline{E}' \arrow[d, "P'"] \\
\underline{B} \arrow[r, bend left, "f_{\perp}"] \arrow[r, bend right, "g_{\perp}"'] \arrow[r, phantom, "\Downarrow \alpha_{\perp}"] & \underline{B}'
\end{tikzcd}$$

Commutative cylinder.

$\underline{cFib} \subseteq \underline{Fib}$ is the locally full sub-2-category where we require the arrows f^T to preserve the cleavages.

We obtain two equivalences of 2-categories:

- $\underline{Fib} \simeq \underline{ICat}$

$$\begin{tikzcd} \underline{\mathcal{B}} \arrow[dd] \arrow[dr] & \\ {} \arrow[r, phantom, "\Downarrow"] & \underline{Cat} \\ \underline{\mathcal{B}}' \arrow[ur] & \end{tikzcd}$$

pseudo natural transf^{ns}

- $\underline{cFib} \simeq \underline{ICat}_s$

$\underline{\text{strict}}$ ntl. transformations

Examples of Fibrations

- $\underline{\text{Mod}} \longrightarrow \underline{\text{Ring}}$ is a fibration (and an opfibration)

  $(R, M) \longmapsto R$

  M a left R-module

  $(R_1, f^* M)$ $\quad$ (R_2, M)

  $R_1 \xrightarrow[f]{} R_2$

  $f^* M$: restriction of scalars

- Codomain fibration over a category $\underline{C}$ with pullbacks:

  $\text{Arr}_s(\underline{C}) = \underline{C}^2$ has obj. $\begin{array}{c} x \\ {\scriptstyle f}\downarrow \\ y \end{array}$ in $\underline{C}$

  and arrows:

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ Y \arrow[r, "k"] & Y' \end{tikzcd}$$

  comm. in $\underline{C}$.

the codomain functor $\underline{C}^2 \longrightarrow \underline{C}$ is a fibration and opfibration; cartesian arrows are pullback squares.

Examples (continued)

- for $\underline{C}$ any category, $\mathrm{Fam}(\underline{C})$ is the category of set-indexed families of obj.^{s} in $\underline{C}$:

$$(C_i)_{i \in I}$$

morphisms:

$$(C_i)_{i\in I} \xrightarrow[(f, (\phi_i)_{i\in I})]{} (D_j)_{j\in J}$$

$f : I \longrightarrow J$ function

$\phi_i : C_i \longrightarrow D_{f(i)}$

This is the Grothendieck construction for

$$\underline{\mathrm{Set}}^{op} \longrightarrow \underline{\mathrm{Cat}}$$
$$I \longmapsto \prod_{i\in I} \underline{C}$$

so we get that $\mathrm{Fam}(\underline{C}) \to \underline{\mathrm{Set}}$ is a fibration

The Category of Elements is also an oplax colimit for F as diagram in <u>Cat</u>:

- we have a universal cone:

$$\begin{tikzcd} A \arrow[dd, "g"'] \\ \\ B \end{tikzcd} \qquad \begin{tikzcd} FA \arrow[rr, "\varepsilon_A"] & & \int_{\mathcal{A}} F \\ & \Downarrow \varepsilon_g & \\ FB \arrow[uu, "Fg"] \arrow[uurr, "\varepsilon_B"'] & & \end{tikzcd}$$

$$\varepsilon_A : \quad x \longmapsto (A, x)$$

$$(\psi : x \to y) \longmapsto \left[(A,x) \xrightarrow[(1_A,\ \varphi_A \circ \psi)]{} (A,y) \right]$$

$$(\varepsilon_g)_y = (g, 1_{Fg(y)}) : (A, Fg(y)) \longrightarrow (B, y)$$

Other properties of the category of elements

- a functor $P : \mathcal{A}^{op} \longrightarrow \underline{\mathrm{Set}}$ is representable (i.e. $P \cong \mathrm{Hom}(\_, X)$ for some X in $\mathcal{A}$) iff $\int_{\mathcal{A}} P$ has a terminal object.

- We can obtain $\int_{\mathcal{A}} P$ as a comma square (see nlab)

ACT Tutorial

on Double Fibrations

Double Grothendieck

Double Colimits

Part II

Goal : Introduce a Double Category of Elements

- with a corresponding notion of double fibration
- extending the monoidal Grothendieck construction [Moeller, Vasilakopoulou, TAC 2020]
- Jaz-Myers' double Grothendieck constructions for open dynamical systems as special cases [EPTC 2021]
- structured and decorated cospans as special cases. [Baez-Courser-Vasilakopoulou, 2020, 2022] [Patterson, 2023]
- extending the discrete case [Lambert, TAC 2021]