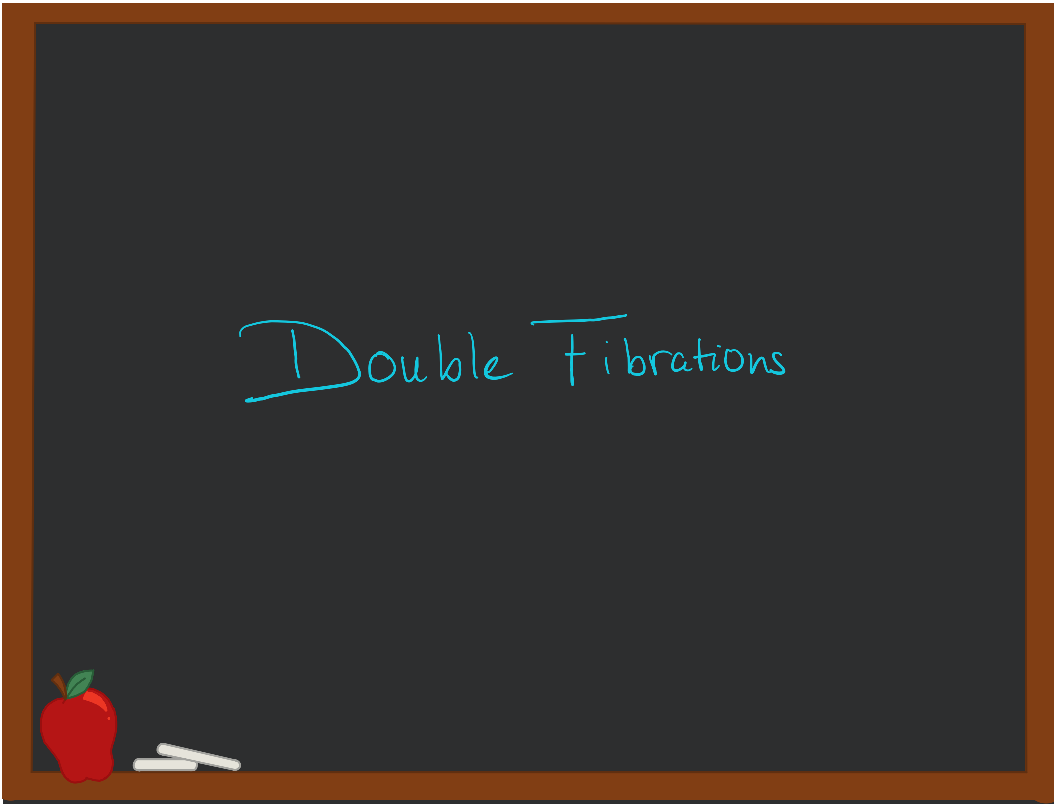

Double Fibrations

Since it is not obvious what the codomain of an indexing functor

$$\mathbb{A} \longrightarrow ?? \qquad (\mathbb{A} \text{ dbl cat}^{y})$$

should be, we start with double fibrations.

Double Fibrations

First Idea: a double category is a pseudo cat$^{\text{d}}$ in Cat $\rightsquigarrow$ a double fibration is a pseudo cat$^{\text{d}}$ in Fib.

This would amount to:

$$
\begin{tikzcd}
E_1 \times_{E_0} E_1 \arrow[r, "\otimes^{\top}"] \arrow[d, "P_1 \times_{P_0} P_1"'] & E_1 \arrow[r, shift left=3, "s^{\top}"] \arrow[r, shift right=3, "t^{\top}"'] \arrow[d, "P_1"] & E_0 \arrow[l, "y^{\top}"'] \arrow[d, "P_0"] \\
B_1 \times_{B_0} B_1 \arrow[r, "\otimes_{\perp}"] & B_1 \arrow[r, shift left=3, "s^{\perp}"] \arrow[r, shift right=3, "t^{\perp}"'] & B_0 \arrow[l, "y^{\perp}"']
\end{tikzcd}
$$

$\rightsquigarrow$ strict dbl functor with extra properties?

Double fibrations

Problem: $\mathbf{Fib}$ doesn't have all 2-pullbacks required for this.

Observation:

- We require the same fibrational strictness for s and t that we require for y and $\otimes$.

Solution: We will require that s and t are in $c\mathbf{Fib}$.

(i.e., they preserve cleavages)

Double Fibrations

Definition: A <u>double fibration</u> is a (strict) double functor $P: \mathbb{E} \to \mathbb{B}$ between pseudo double categories:

$$
\begin{tikzcd}
\underline{E}_1 \times_{\underline{E}_0} \underline{E}_1 \arrow[r, "\otimes_E"] \arrow[d, "P_1 \times_{P_0} P_1"'] & \underline{E}_1 \arrow[r, shift left=2, "s_E"] \arrow[r, shift right=2, "t_E"'] \arrow[d, "P_1"'] & \underline{E}_0 \arrow[l] \arrow[d, "P_0"] \\
\underline{B}_1 \times_{\underline{B}_0} \underline{B}_1 \arrow[r, "\otimes_B"] & \underline{B}_1 \arrow[r, shift left=2, "s_B"] \arrow[r, shift right=2, "t_B"'] & \underline{B}_0 \arrow[l]
\end{tikzcd}
$$

s.t.

- P_0 and P_1 are fibrations with cleavages
- s_E and t_E are cleavage preserving
- y_E and $\otimes_E$ are cartesian-morphism preserving.

Examples

1. $\mathrm{Im} : \mathbb{S}\mathrm{pan}(\underline{\mathrm{Set}}) \longrightarrow \mathbb{R}\mathrm{el}$

$$(A \xleftarrow{s} S \xrightarrow{t} B) \longmapsto (\mathrm{Im}((s,t)) \subseteq A \times B)$$

is a double opfibration.

2. When $\underline{E}_0 = \underline{B}_0 = 1$, we get:

$$\begin{tikzcd}
\underline{E}_1 \times_{\underline{E}_0} \underline{E}_1 \arrow[r, "\otimes_E"] \arrow[d, "P_1 \times_{P_0} P_1"'] & \underline{E}_1 \arrow[d, "P_1"'] & 1 \arrow[l, "y"'] \arrow[d, "id"] \\
\underline{B}_1 \times_{\underline{B}_0} \underline{B}_1 \arrow[r, "\otimes_B"] & \underline{B}_1 & 1 \arrow[l, "y"']
\end{tikzcd}$$

a monoidal fibration!

3. The Grothendieck constructions given in David Jaz Myers' work are also double fibrations.

Examples

4. For any 2-functor $P : \underline{E} \longrightarrow \underline{B}$, P is a 2-fibration as in Buckley's work if and only if $\mathbb{Q}(P) : \mathbb{Q}(\underline{E}) \longrightarrow \mathbb{Q}(\underline{B})$ is a double fibration.

5. if P_0 and P_1 are discrete fibrations, we recover discrete double fibrations.

6. For $\mathbb{D}$ a double cat^y, let $\mathbb{D}^2 = \begin{pmatrix} \mathbb{D}_1^2 \\ \downarrow\downarrow \\ \mathbb{D}_0^2 \end{pmatrix}$

$$\mathrm{dom} : \mathbb{D}^2 \longrightarrow \mathbb{D}$$

is a double fibration

7. The codomain fibration extends to a double codomain fibration $\mathrm{cod} : \mathbb{D}^2 \to \mathbb{D}$ if:

- $\mathbb{D}_1$ and $\mathbb{D}_0$ have chosen finite limits
- these limits are preserved on the nose by s and t
- and up to iso by y and $\otimes$.

8. for $\mathcal{C}$ a small cat^y, $\mathrm{Fam}(\mathcal{C})$ has:

<u>obj</u>: $f : I \longrightarrow \mathcal{C}$, or $(I, \{C_i\}_{i \in I})$

<u>arrows</u>: $\dfrac{(h, \alpha) : f \longrightarrow g}{h : I \longrightarrow J, \quad \alpha : f \Rightarrow gh}$

$$\begin{tikzcd} I \arrow[rr, "h"] \arrow[dr, "f"'] & \arrow[d, phantom, "\overset{\Rightarrow}{\alpha}"] & J \arrow[dl, "g"] \\ & \mathcal{C} & \end{tikzcd}$$

<u>or</u>:

$$\begin{tikzcd} (I, \{C_i\}_{i \in I}) \arrow[d, "{(h : I \to J, \{\alpha_i : C_i \to C_{h(i)}\}_{i \in I})}"] \\ (J, \{C_j\}_{j \in J}) \end{tikzcd}$$

pro arrows : $f \xrightarrow{(S,\, d_0,\, d_1,\, \theta)}$

for natural transformations

$$
\begin{tikzcd}
S \arrow[r, "d_1"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\overset{\theta}{\Rightarrow}"] & K \arrow[d, "p"] \\
I \arrow[r, "f"'] & e
\end{tikzcd}
$$

for a span of functions $I \xleftarrow{d_0} S \xrightarrow{d_1} K$

or :

$$(I, \{C_i\}_{i \in I}) \xrightarrow[((d_0, S, d_1), \theta_s)]{} (K, \{C_k\})$$

for a span of functions $I \xleftarrow{d_0} S \xrightarrow{d_1} K$

+ a family of arrow

$$\theta_s : C_{d_0(s)} \longrightarrow C_{d_1(s)}$$

cells: a cell from

$$\begin{tikzcd} S \arrow[r, "d_1"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\overset{\theta}{\Rightarrow}"] & K \arrow[d, "p"] \\ I \arrow[r, "f"'] & e \end{tikzcd} \quad \text{to} \quad \begin{tikzcd} T \arrow[r, "d_1'"] \arrow[d, "d_0'"'] \arrow[dr, phantom, "\overset{\delta}{\Rightarrow}"] & L \arrow[d, "q"] \\ J \arrow[r, "g"'] & e \end{tikzcd}$$

is given by a morphism of spans:

$$\begin{tikzcd} I \arrow[d, "h"'] & S \arrow[l, "d_0"'] \arrow[r, "d_1"] \arrow[d, "m"] & K \arrow[d, "r"] \\ J & T \arrow[l, "d_0'"] \arrow[r, "d_1"'] & L \end{tikzcd}$$

with 2-cells:

$$\begin{tikzcd} & I \arrow[dl, "f"'] \arrow[d, "h"] \\ e \arrow[r, phantom, "\overset{\alpha}{\Rightarrow}"] & J \arrow[l, "g"] \end{tikzcd} \qquad \begin{tikzcd} & K \arrow[dl, "p"'] \arrow[d, "r"] \\ e \arrow[r, phantom, "\overset{\beta}{\Rightarrow}"] & L \arrow[l, "q"] \end{tikzcd}$$

st:

$$(\beta * d_1)\theta = (\delta * q)(\alpha * d_0)$$

Or: a family of cells:

$$
\begin{tikzcd}
(I, \{C_i\}_{i \in I}) \arrow[r, "{((d_0, S, d_1), \theta)}", "/" marking] \arrow[d, "{(h, (\alpha_i))}"'] \arrow[dr, phantom, "m"] & (K, \{C_k\}_{k \in K}) \arrow[d, "{(r, (\beta_k))}"] \\
(J, \{C_j\}_{j \in J}) \arrow[r, "{((d_0', T, d_1'), \theta')}"', "/" marking] & (L, \{C_\ell\}_{\ell \in L})
\end{tikzcd}
$$

when $m : S \to T$ fits in

$$
\begin{tikzcd}
I \arrow[d, "h"'] & S \arrow[l, "d_0"'] \arrow[r, "d_1"] \arrow[d, "m"] \arrow[dl, phantom, "="] \arrow[dr, phantom, "\Nearrow"] & K \arrow[d, "r"] \\
J & T \arrow[l, "d_0'"] \arrow[r, "d_1'"'] & L
\end{tikzcd}
$$

and we require that for each $s \in S$:

$$\begin{tikzcd}
C_{d_0(s)} \arrow[r, "\theta_s"] \arrow[d, "\alpha_{d_0(s)}"'] & C_{d_1(s)} \arrow[d, "\beta_{d_1(s)}"] \\
C_{h(d_0 s)} \arrow[r, "\theta'_{m(s)}"] & C_{r(d_1 s)}
\end{tikzcd}$$

this square is well defined because

$$d_0'(m(s)) = h\, d_0(s)$$

and $\quad d_1'(m(s)) = r\, d_1(s)$

We only need to require that

$$\beta_{d_1(s)}\, \theta_s = \theta'_{m(s)} \circ \alpha_{d_0(s)},$$

because $\mathcal{C}$ has no 2-cells.

Note: this can be extended to $\mathcal{C}$ a dbl. cat^y

$$\Pi_0 : \mathbb{F}\mathrm{am}(\mathcal{C})_0 \longrightarrow \underline{\mathrm{Set}}$$

is a split fibration.

We extend this to

$$\Pi : \mathbb{F}\mathrm{am}(\mathcal{C}) \longrightarrow \mathrm{Span}(\underline{\mathrm{Set}}).$$

(send proarrows to their underlying spans and cells to span morphisms)

<u>Claim:</u> this is a split double fibration.

Relation with Street's Internal Fibrations

(a look-off trail we may not take)

Internal Fibrations

- Internal fibrations in a 2-category were introduced by Street in 1974.
- We will use the following three 2-cat's:

$$\underbrace{\mathbf{Dbl}_s}_{\text{strict dbl functors}} \subseteq \underbrace{\mathbf{Dbl}}_{\text{pseudo dbl functors}} \subseteq \underbrace{\mathbf{Dbl}_\ell}_{\text{lax dbl functors}}$$

Say more about Street's definition?

- In all 3 cases we use vertical transformations
  $\alpha : F \Rightarrow G : \mathbb{D} \rightrightarrows \mathbb{E}$:
  * for each x in $\mathbb{D}$: an arrow $\alpha_x : Fx \to Gx$ in $\mathbb{E}$
  * for each pro arrow $x \overset{m}{\nrightarrow} y$ a dbl cell $\alpha_m : Fm \to Gm$ in $\mathbb{E}_1$. (natural and functorial)

Theorem (Cruttwell, Lambert, P, Szyld)

A strict double functor is an internal fibration in DblCat if and only if it is a double fibration

In addition, a pseudo double functor P

- is an internal fibration in DblCat_ℓ iff P_0 and P_1 admit cleavages that are preserved by $s_{\mathbb{E}}$ and $t_{\mathbb{E}}$.
- is an internal fibration in DblCat iff, in addition, $y_{\mathbb{E}}$ and $\otimes_{\mathbb{E}}$ preserve Cartesian morphisms.

Furthermore, a <u>strict</u> double functor P is an internal fibration in $\underline{\mathbf{DblCat}}_s$ if and only if P_0 and P_1 are fibrations that admit cleavages that are preserved by all of $s_{\mathbb{E}}$, $t_{\mathbb{E}}$, $y_{\mathbb{E}}$ and $\otimes_{\mathbb{E}}$

Double Indexing Functors

(Take 1)

Double Indexing Functors

Exercise!

Note: (1.) Categories are monoids in Span(Set)

$$
\begin{tikzcd}
C_0 \arrow[d, equal] & C_0 \arrow[l, "id"'] \arrow[r, "id"] \arrow[d, "y"] & C_0 \arrow[d, equal] \\
C_0 & C_1 \arrow[l, "s"] \arrow[r, "t"'] & C_0
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & C_1 \times_{C_0} C_1 \arrow[dl] \arrow[dr] \arrow[dd, bend right, "\mu"] & & \\
C_0 \arrow[d, equal] & C_1 \arrow[l, "s"'] \arrow[r, "t"] & C_0 & C_1 \arrow[l, "s"'] \arrow[r, "t"] & C_0 \arrow[d, equal] \\
C_0 & & C_1 \arrow[ll, "s"] \arrow[rr, "t"'] & & C_0
\end{tikzcd}
$$

Unit $\longleftrightarrow$ identities for the cat^y

multiplication $\longleftrightarrow$ composition for the cat^y.

(2) Moeller and Vasilakopoulou used:

$$\underline{\mathrm{Fib}} \simeq \underline{\mathrm{ICat}} \Longrightarrow \underline{\mathrm{Ps\,Mon}}(\underline{\mathrm{Fib}}, \times) \simeq \underline{\mathrm{Ps\,Mon}}(\underline{\mathrm{ICat}}, \times)$$

To generalize this further we need:

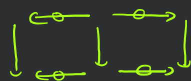
* double 2-categories (pseudo category objects in $\underline{2\text{-}\mathcal{C}at}$)

* pseudo monoids in double 2-categories

$\underline{\text{Result}}$: pseudo categories in a 2-cat^y $\mathcal{C}$ correspond to pseudo monoids in $\mathcal{S}pan(\mathcal{C})$.

Recall that we want to take the source and target from a more restricted class of arrows, say Σ:

$\underline{\text{Result}}$: pseudo cat^s in $\mathcal{C}$ with s, t in Σ correspond to pseudo monoids in $\mathcal{S}pan_{\Sigma}(\mathcal{C})$.



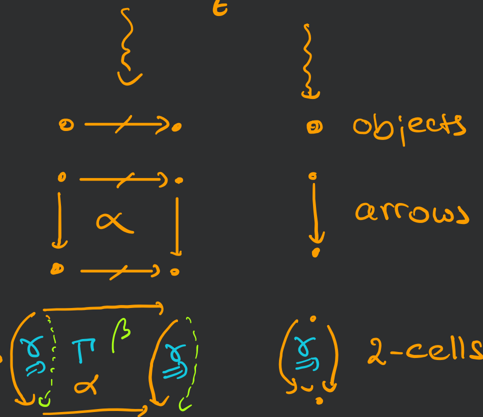
Double 2-Categories

A double 2-category is a pseudo cat^y in <u>2Cat</u>

$$\mathcal{E}_1 \times_{\mathcal{E}_0} \mathcal{E}_1 \xrightarrow{\otimes} \mathcal{E}_1 \underset{t}{\overset{s}{\rightrightarrows}} \mathcal{E}_0 \quad (i : \mathcal{E}_0 \to \mathcal{E}_1)$$

so we have

$\Gamma : \alpha \Rightarrow \beta$



The Double 2-Category $\mathbb{S}\mathrm{pan}(\underline{\mathrm{Cat}})$

<u>Objects</u> categories $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$

<u>Arrows</u> functors

<u>Proarrows</u> Spans $\mathcal{S} \xleftarrow{S} \mathcal{A} \xrightarrow{T} \mathcal{T}$

<u>Dbl cells</u> Commutative diagrams

$$\begin{tikzcd}
\mathcal{S} \arrow[d, "G"'] & \mathcal{A} \arrow[l, "S"'] \arrow[r, "T"] \arrow[d, "F"] & \mathcal{T} \arrow[d, "H"] \\
\mathcal{S}' & \mathcal{A}' \arrow[l, "S'"'] \arrow[r, "T'"] & \mathcal{T}'
\end{tikzcd}$$

Dbl 3-cells

$$\begin{tikzcd}
\mathcal{S} \arrow[d, "\mathcal{G}"'] & \mathcal{A} \arrow[l, "S"'] \arrow[r, "T"] \arrow[d, "F"] & \mathcal{T} \arrow[d, "H"] \\
\mathcal{X} & \mathcal{B} \arrow[l, "X"] \arrow[r, "Y"'] & \mathcal{Y}
\end{tikzcd}
\qquad (\gamma, \varphi, \theta) \Rrightarrow
\qquad
\begin{tikzcd}
\mathcal{S} \arrow[d, "\mathcal{G}'"'] & \mathcal{A} \arrow[l, "S"'] \arrow[r, "T"] \arrow[d, "F'"] & \mathcal{T} \arrow[d, "H'"] \\
\mathcal{X} & \mathcal{B} \arrow[l, "X"] \arrow[r, "Y"'] & \mathcal{Y}
\end{tikzcd}$$

Consists of

$$\begin{tikzcd}
\mathcal{S} \arrow[d, bend right=50, "G"'] \arrow[d, bend left=50, dashed, "G'"] \arrow[d, phantom, "\overset{\gamma}{\Rightarrow}"] & \mathcal{A} \arrow[l, "S"'] \arrow[r, "T"] \arrow[d, bend right=50, "F"'] \arrow[d, bend left=50, dashed, "F'"] \arrow[d, phantom, "\overset{\varphi}{\Rightarrow}"] & \mathcal{T} \arrow[d, bend right=50, "H"'] \arrow[d, bend left=50, "H'"] \arrow[d, phantom, "\overset{\theta}{\Rightarrow}"] \\
\mathcal{X} & \mathcal{B} \arrow[l, "X"] \arrow[r, "Y"'] & \mathcal{Y}
\end{tikzcd}$$

such that the two cylinders commute.

Double Indexing Functors

$\mathbb{I}\mathbb{S}\mathrm{pan}(\underline{\mathrm{Cat}})$ is the slice category with

- objects: Contravariant lax double pseudo functors $\mathcal{F}: \mathbb{A}^{op} \longrightarrow \mathbb{S}\mathrm{pan}(\underline{\mathrm{Cat}})$ ($\mathbb{A}$ - indexed ps. dbl. categories)
- morphisms: $(H, \tau): \mathcal{F} \longrightarrow \mathcal{G}$:

H lax dbl. pseudo functor

τ: lax dbl. ps. ntl. trafo

$$\begin{tikzcd} \mathbb{A}^{op} \arrow[rr, "H^{op}"] \arrow[dr, "\mathcal{F}"'] & \arrow[d, phantom, "\overset{\tau}{\Rightarrow}"] & \mathbb{B}^{op} \arrow[dl, "\mathcal{G}"] \\ & \mathbb{S}\mathrm{pan}(\underline{\mathrm{Cat}}) & \end{tikzcd}$$

Lax Double Pseudo Functors

Definition A lax double pseudo functor

$$F : \mathbb{D} \longrightarrow \mathbb{E}$$

between dbl 2-categories $\mathbb{D}, \mathbb{E}$, consists of:

- Pseudo functors: $F_0 : \mathbb{D}_0 \longrightarrow \mathbb{E}_0$ (now 2-categories!)
  $F_1 : \mathbb{D}_1 \longrightarrow \mathbb{E}_1$ pseudo in the arrow direction!
- Comparison pseudo natural transformations:

$$\begin{tikzcd}
\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \arrow[r, "\otimes"] \arrow[d, "F_1 \times F_1"'] \arrow[dr, phantom, "\overset{\phi}{\Longrightarrow}"] & \mathbb{D}_1 \arrow[d, "F_1"] \\
\mathbb{E}_1 \times_{\mathbb{E}_0} \mathbb{E}_1 \arrow[r, "\otimes"'] & \mathbb{E}_1
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{D}_0 \arrow[r, "y"] \arrow[d, "F_0"'] \arrow[dr, phantom, "\overset{\iota}{\Longrightarrow}"] & \mathbb{D}_1 \arrow[d, "F_1"] \\
\mathbb{E}_0 \arrow[r, "y"'] & \mathbb{E}_1
\end{tikzcd}$$

lax in the proarrow direction

- Invertible associativity and unitor modifications

$$
\begin{tikzcd}
& \mathbb{D}_1^{(3)} \arrow[dl, "F_1^{(3)}"'] \arrow[dr, "\otimes\times 1"] \arrow[dd] & \\
\mathbb{E}_1^{(3)} \arrow[dd, "1\times\otimes"'] \arrow[r, phantom, "1\times\phi"] & {} \arrow[r, phantom, "\overset{a}{\cong}"] & \mathbb{D}_1^{(2)} \arrow[dd, "\otimes"] \\
& \mathbb{D}_1^{(2)} \arrow[dl] \arrow[dr] & \\
\mathbb{E}_1^{(2)} \arrow[dr, "\otimes"'] & {} \arrow[d, phantom, "\overset{\phi}{\Longrightarrow}"] & \mathbb{D}_1 \arrow[dl, "F_1"] \\
& \mathbb{E}_1 &
\end{tikzcd}
\quad \overset{\Phi}{\cong} \quad
\begin{tikzcd}
& \mathbb{D}_1^{(3)} \arrow[dl, "F_1^{(3)}"'] \arrow[dr, "\otimes\times 1"] \arrow[d, phantom, "\overset{\phi\times 1}{\Longrightarrow}"] & \\
\mathbb{E}_1^{(3)} \arrow[dr] \arrow[dd, "1\times\otimes"'] & {} & \mathbb{D}_1^{(2)} \arrow[dl] \arrow[dd, "\otimes"] \\
{} \arrow[r, phantom, "\overset{a}{\cong}"] & \mathbb{E}_1^{(2)} \arrow[dd] \arrow[r, phantom, "\overset{\phi}{\Longrightarrow}"] & {} \\
\mathbb{E}_1^{(2)} \arrow[dr, "\otimes"'] & & \mathbb{D}_1 \arrow[dl, "F_1"] \\
& \mathbb{E}_1 &
\end{tikzcd}
$$

etc.

satisfying well-definedness and coherence conditions.

Lax Double Pseudo Natural Transformations

A lax dbl ps. ntl transformation:

$$\tau : F \Rightarrow G : \mathbb{D} \rightrightarrows \mathbb{E}$$

Consists of:

- pseudo ntl transformations

$$\tau_0 : F_0 \Rightarrow G_0, \quad \tau_1 : F_1 \Rightarrow G_1$$

(arrow-component transformations)

- modifications:

$$\begin{tikzcd}
\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \arrow[r, equal] \arrow[d, "F_1 \times_{F_0} F_1"'] \arrow[dr, phantom, "\overset{\tau_1 \times_{\tau_0} \tau_1}{\Rightarrow}"] & \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \arrow[r, "\otimes"] \arrow[d] & \mathbb{D}_1 \arrow[d, "G_1"] \arrow[r, phantom, ""] & \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \arrow[r, "\otimes"] \arrow[d, "F_1 \times_{F_0} F_1"'] \arrow[dr, phantom, "\overset{\phi}{\Rightarrow}"] & \mathbb{D}_1 \arrow[r, equal] \arrow[d, "F_1"'] \arrow[dr, phantom, "\overset{\tau_1}{\Rightarrow}"] & \mathbb{D}_1 \arrow[d, "G_1"] \\
\mathbb{E}_1 \times_{\mathbb{E}_0} \mathbb{E}_1 \arrow[r, equal] & \mathbb{E}_1 \times_{\mathbb{E}_0} \mathbb{E}_1 \arrow[r, "\otimes"] & \mathbb{E}_1 \arrow[u, phantom, shift left=8, "\overset{T}{\Rrightarrow}"] & \mathbb{E}_1 \times_{\mathbb{E}_0} \mathbb{E}_1 \arrow[r, "\otimes"'] & \mathbb{E}_1 \arrow[r, equal] & \mathbb{E}_1
\end{tikzcd}$$

and

$$
\begin{tikzcd}
\mathbb{D}_0 \arrow[r, equal] \arrow[d, "F_0"'] \arrow[dr, phantom, "\overset{\tau_0}{\Rightarrow}"] & \mathbb{D}_0 \arrow[r, "y"] \arrow[d] \arrow[dr, phantom, "\overset{\cong}{\Rightarrow}"] & \mathbb{D}_1 \arrow[d, "G_1"] \\
\mathbb{E}_0 \arrow[r, equal] & \mathbb{E}_0 \arrow[r, "y"'] & \mathbb{E}_1
\end{tikzcd}
\overset{I}{\Rrightarrow}
\begin{tikzcd}
\mathbb{D}_0 \arrow[r, "y"] \arrow[d, "F_0"] \arrow[dr, phantom, "\overset{\cong}{\Rightarrow}"] & \mathbb{D}_1 \arrow[r, equal] \arrow[d, "F_1"'] \arrow[dr, phantom, "\overset{\tau_1}{\Rightarrow}"] & \mathbb{D}_1 \arrow[d, "G_1"] \\
\mathbb{E}_0 \arrow[r, "y"'] & \mathbb{E}_1 \arrow[r, equal] & \mathbb{E}_1
\end{tikzcd}
$$

satisfying multiplicativity and unitality conditions.

We write $\underline{\mathrm{Dbl2Cat}}(\mathbb{D}, \mathbb{E})$ for the cat^y of lax dbl pseudofunctors and lax dbl. ps. ntl. transformations.

The Representation Theorem

The Representation Theorem

<u>Theorem</u> (Cruttwell, Lambert, P., Szyld)

There is an equivalence of categories

$$\underline{\mathrm{DblFib}} \simeq \mathrm{I}\,\mathrm{Span}(\underline{\mathrm{Cat}})$$

<u>Idea for the proof:</u> use pseudo monoids in double 2-categories.

$$\underline{\mathrm{Fib}} \simeq \underline{\mathrm{ICat}} \rightsquigarrow \underline{\mathrm{cFib}} \simeq \underline{\mathrm{ICat}}_t$$

so: $\mathrm{Span}_c(\underline{\mathrm{Fib}}) \simeq \mathrm{Span}_t(\underline{\mathrm{ICat}})$

(Convince yourself!)

$\mathrm{Span}_c(\underline{\mathrm{Fib}})$ has objects:
$$\begin{tikzcd} \underline{E} \arrow[d, "p"] \\ \underline{B} \end{tikzcd}$$
fibrations with cleavage

Notation:

P

$\mathrm{Span}_t(\underline{\mathrm{ICat}})$ has objects: $\underline{B}^{op} \xrightarrow{F} \underline{\mathrm{Cat}}$ ps. functor

F

arrows:
$$\begin{tikzcd} \underline{E} \arrow[r, "f^{\top}"] \arrow[d, "p"'] & \underline{E}' \arrow[d, "p'"] \\ \underline{B} \arrow[r, "f^{\perp}"'] & \underline{B}' \end{tikzcd}$$
cartesian arrow preserving

$$\begin{tikzcd} P \arrow[d, "f"] \\ P' \end{tikzcd}$$

arrows:
$$\begin{tikzcd} \underline{B}^{op} \arrow[rr, "H"] \arrow[dr, "F"'] & \arrow[d, phantom, "\overset{\theta}{\Rightarrow}"] & (\underline{B}')^{op} \arrow[dl, "F'"] \\ & \underline{\mathrm{Cat}} & \end{tikzcd}$$
θ pseudo trafo

$$\begin{tikzcd} F \arrow[d, "{(H,\theta)}"'] \\ F' \end{tikzcd}$$

Proarrows : $P \xleftarrow{\ell} Q \xrightarrow{r} R$

ℓ^T, r^T cleavage preserving

Proarrows : $F \xleftarrow{(L,\lambda)} \mathcal{G} \xrightarrow{(R,\rho)} K$

λ, ρ strict natural transf^{ns}

Cells :

$$\begin{tikzcd}
P \arrow[d, "f"'] & Q \arrow[l, "\ell"'] \arrow[r, "r"] \arrow[d, "g"] & R \arrow[d, "h"] \\
P' & Q' \arrow[l, "\ell'"] \arrow[r, "r'"'] & R'
\end{tikzcd}$$

$\ell^T, r^T, \ell'^T, r'^T$ cleavage preserving

f^T, g^T, h^T Cartesian arrow pres.

Cells:

$$\begin{tikzcd}
F \arrow[d, "{(H,\theta)}"'] & \mathcal{G} \arrow[l, "{(L,\lambda)}"'] \arrow[r, "{(R,\rho)}"] \arrow[d] & K \arrow[d] \\
F' & \mathcal{G}' \arrow[l, "{(L',\lambda')}"] \arrow[r, "{(R',\rho')}"'] & K'
\end{tikzcd}$$

Then lift :

$$\underline{\mathrm{Dbl}\,\mathrm{Fib}} := \underline{\mathrm{Ps}\,\mathrm{Mon}}\left(\mathbb{S}\mathrm{pan}_c(\underline{\mathrm{Fib}})\right) \simeq$$

$$\simeq \underline{\mathrm{Ps}\,\mathrm{Mon}}\left(\mathbb{S}\mathrm{pan}_t(\mathbb{I}\underline{\mathrm{Cat}})\right) \simeq \mathbb{I}\mathbb{S}\mathrm{pan}(\underline{\mathrm{Cat}}).$$

Connections with Known Constructions

$$
\begin{tikzcd}
\text{Monoidal Fibrations} \arrow[dr, "\text{monoids are cats}"'] & & \text{2-Fibrations} \arrow[dl, "\text{quintet constr.}"] \\
& \text{Double Fibrations} & \\
\text{Discrete Dbl Fib.} \arrow[ur] & & \text{Double Gr. Constr.} \arrow[ul]
\end{tikzcd}
$$

All of these have an indexed notion and a Grothendieck category of elements.

In our case the composition of the functors used in the proof gives us also a category of elements construction.

The Double Category of Elements

The Double Grothendieck Construction

Start with $F : \mathbb{D}^{op} \longrightarrow \mathrm{Span}(\underline{\mathrm{Cat}})$:

$$F_0 : \mathbb{D}_0^{op} \longrightarrow \mathrm{Span}(\underline{\mathrm{Cat}})_0 = \underline{\mathrm{Cat}} \qquad (1)$$

$$F_1 : \mathbb{D}_1^{op} \longrightarrow \mathrm{Span}(\underline{\mathrm{Cat}})_1$$

and a further induced functor:

$$\mathbb{D}_1^{op} \xrightarrow{F_1} \mathrm{Span}(\underline{\mathrm{Cat}})_1 \xrightarrow{apx} \underline{\mathrm{Cat}} \qquad (2)$$

Apply the ordinary elements construction to (1) and (2):

$$\mathrm{El}(F)_0 \longrightarrow \mathbb{D}_0 \qquad \mathrm{El}(F)_1 \longrightarrow \mathbb{D}_1$$

cloven fibrations.

Details of $\mathbb{E}\mathrm{l}(F)$

Some notation:

- for $m : A \nrightarrow B$ in $\mathbb{D}$, write Fm in $\mathbb{S}\mathrm{pan}(\underline{\mathrm{Cat}})$ as:

$$\begin{tikzcd} & Fm \arrow[dl, "L_m"'] \arrow[dr, "R_m"] & \\ FA & & FB \end{tikzcd}$$

- for $A \overset{m}{\nrightarrow} B \overset{n}{\nrightarrow} C$ in $\mathbb{D}$, we have the laxity comparison cell

$$\begin{tikzcd} FA \arrow[d, equal] & Fm \underset{FB}{\times} Fn \arrow[l] \arrow[r] \arrow[d, "\phi_{m,n}"] & FC \arrow[d, equal] \\ FA & F(m \otimes n) \arrow[l, "L_{m\otimes n}"'] \arrow[r, "R_{m\otimes n}"] & FC \end{tikzcd}$$

Now $\mathbb{E}l(F)$ is given by:

- objects: (C, x) $\quad$ C in $\mathbb{D}$, x in FC.

- arrows: $(f, \bar{f}) : (C, x) \longrightarrow (D, y)$

  with $f : C \to D$ in $\mathbb{D}$

  and $\bar{f} : x \to f^* y \ \left(= F(f)(y)\right)$ in FC.

- proarrows: $(m, \bar{m}) : (C, x) \nrightarrow (D, y)$

  with $C \overset{m}{\nrightarrow} D$ in $\mathbb{D}$

  $\bar{m} \in Fm$ $\quad \left(FC \xleftarrow[L_m]{} Fm \xrightarrow[R_m]{} FD \right)$

  s.t. $L_m(\bar{m}) = x$

  $R_m(\bar{m}) = y$

- double cells :

$$\begin{tikzcd}
(A,x) \arrow[r, "{(m,\overline{m})}", "|" marking] \arrow[d, "{(f,\overline{f})}"'] \arrow[dr, phantom, "{(\theta,\overline{\theta})}"] & (B,y) \arrow[d, "{(g,\overline{g})}"] \\
(C,z) \arrow[r, "{(n,\overline{n})}"', "|" marking] & (D,w)
\end{tikzcd}$$

with

$$\begin{tikzcd}
A \arrow[r, "m", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\theta"] & B \arrow[d, "g"] \\
C \arrow[r, "n"', "|" marking] & D
\end{tikzcd}
\quad \text{in } \mathbb{D} \quad
\begin{tikzcd}
FA \arrow[d, "Ff"'] & Fm \arrow[l, "L_m"'] \arrow[r, "R_m"] \arrow[d, "F\theta"] & FB \arrow[d, "Fg"] \\
FC & Fn \arrow[l, "L_n"] \arrow[r, "R_n"'] & FD
\end{tikzcd}$$

and $\overline{m} \xrightarrow{\overline{\theta}} \theta^{*}\overline{n}$ an arrow in Fm s.t.

$$L_m(\overline{\theta}) = \overline{f} \quad \text{and} \quad R_m(\overline{\theta}) = \overline{g}.$$

Composition in the "arrow direction" is as expected:

- for arrows $(A,x) \xrightarrow{(f,\bar{f})} (B,y) \xrightarrow{(g,\bar{g})} (C,z)$

  the composite is $\left(gf,\ \phi_{f,g}\, f^*(\bar{g})\, \bar{f}\right) : (A,x) \longrightarrow (C,z)$

- for cells $(m,\bar{m}) \xRightarrow{(\theta,\bar{\theta})} (n,\bar{n}) \xRightarrow{(\delta,\bar{\delta})} (p,\bar{p})$

  the composite is

  $$\left(\delta\theta,\ \phi_{\theta,\delta}\, \theta^*(\bar{\delta})\bar{\theta}\right) : (m,\bar{m}) \Longrightarrow (p,\bar{p}).$$

- Units: $\left(1_C, (\varphi_C)_x\right) : (C,x) \longrightarrow (C,x)$

  $\left(1_m,\ (\varphi_m)_{\bar{m}}\right) : (m,\bar{m}) \longrightarrow (m,\bar{m}).$

$$\mathbb{E}\mathrm{l}(F)_1 \underset{t}{\overset{s}{\rightrightarrows}} \mathbb{E}\mathrm{l}(F)_0$$ are defined by

$$s(\theta,\bar{\theta}) = (f,\bar{f})$$

$$t(\theta,\bar{\theta}) = (g,\bar{g})$$

- proarrow composition:

for $(A,x) \overset{(m,\bar{m})}{\nrightarrow} (B,y) \overset{(n,\bar{n})}{\nrightarrow} (C,z)$

the composite is:

$$(m\otimes n,\ \phi_{m,n}(\bar{m},\bar{n})) : (A,x) \nrightarrow (C,z)$$

- Composition for cells and proarrow units are given using appropriate components of the structure isos related to pseudo naturality.

Results

- $\Pi : \mathbb{E}l(F) \to \mathbb{D}$ is a double fibration.
- This is the object part of an equivalence of categories

$$\underline{\mathrm{DblFib}} \simeq \mathbb{I}\,\mathbb{S}\mathrm{pan}(\underline{\mathrm{Cat}})$$

which specializes to

$$\underline{\mathrm{DblFib}}(\mathbb{B}) \simeq \underline{\mathrm{Dbl2Cat}}(\mathbb{B}^{op}; \mathbb{S}\mathrm{pan}(\underline{\mathrm{Cat}}))$$

for each dbl cat^{y} $\mathbb{B}$.

Examples

- Let $\underline{A}$ be a category with pushouts and $\mathbb{C}\mathrm{sp}(\underline{A})$ the double cat^y with cells:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "u"'] & Z \arrow[d, "w"] & Y \arrow[l] \arrow[d, "v"] \\ X' \arrow[r] & Z' & Y' \arrow[l] \end{tikzcd}$$

- For any lax double functor

$$F : \mathbb{C}\mathrm{sp}(\underline{A}) \to \mathbb{S}\mathrm{pan}(\underline{\mathrm{Cat}})$$

the dbl cat^y of elements $\mathbb{E}\mathrm{l}(F)$ is F-$\mathbb{C}\mathrm{sp}$, the double category of F-decorated cospans. [Patterson, 2023].

This slightly generalizes the decorated cospans from Baez-Courser-Vasilakopoulou.