Lens Exercises

1. What data do you need to define lenses of the following forms? (Where $\mathbb{1} = \{*\}$ is a one element set)

   ⓐ $\begin{pmatrix}\mathbb{1}\\ A^+\end{pmatrix} \leftrightarrows \begin{pmatrix}\mathbb{1}\\ B^+\end{pmatrix}$ ⓑ $\begin{pmatrix}A^-\\ \mathbb{1}\end{pmatrix} \leftrightarrows \begin{pmatrix}B^-\\ \mathbb{1}\end{pmatrix}$

   ⓒ $\begin{pmatrix}\mathbb{1}\\ \mathbb{1}\end{pmatrix} \leftrightarrows \begin{pmatrix}B^-\\ B^+\end{pmatrix}$ ⓓ $\begin{pmatrix}A^-\\ A^+\end{pmatrix} \leftrightarrows \begin{pmatrix}\mathbb{1}\\ \mathbb{1}\end{pmatrix}$

2. Prove that lens composition is unital and associative.

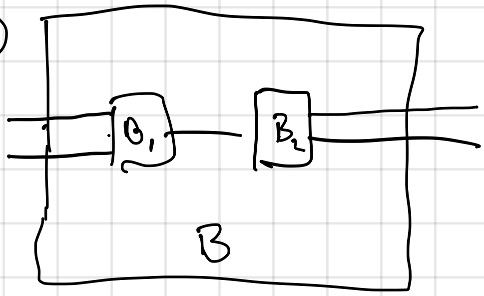
3. ⓐ Show that $\mathbb{1}$ is the monoidal unit for $\otimes$ on lenses.

   ⓑ Is the monoidal product of lenses a cartesian product? Prove it is or give a counterexample.

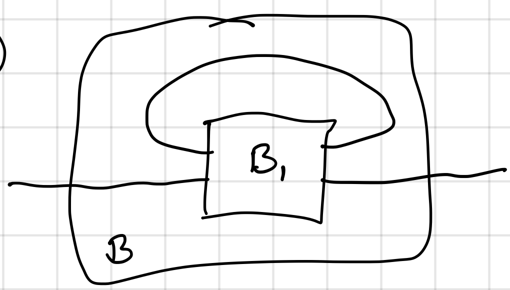
4. Write out lens composition in $\mathbf{Lens}_{\mathcal{C}}$ where $\mathcal{C}$ is an arbitrary cartesian category.

5. Express the following wiring diagrams as lenses in the free cartesian category $\mathbf{Arity} \cong \mathbf{FinSet}^{op}$

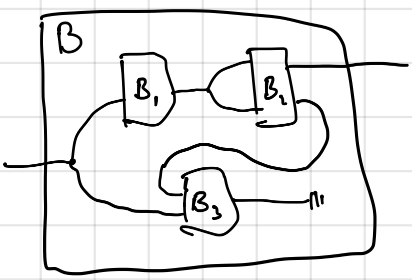
   ⓐ



   ⓑ

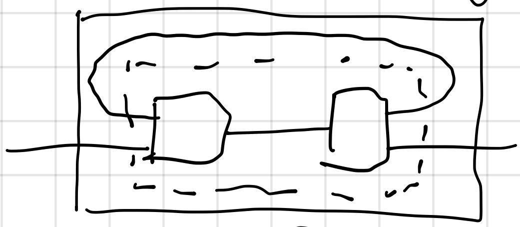


   ⓒ



6. (a) Write out the formula for composing wiring diagrams in terms of lens composition in Arity.

(b) Check that the wiring diagram



is the composite (5b) ∘ (5a).

7. (a) Show that for any map $f : X \to Y$, the square

$$\begin{tikzcd} X \times A \arrow[r, "f \times A"] \arrow[d, "\pi"'] & Y \times A \arrow[d, "\pi"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

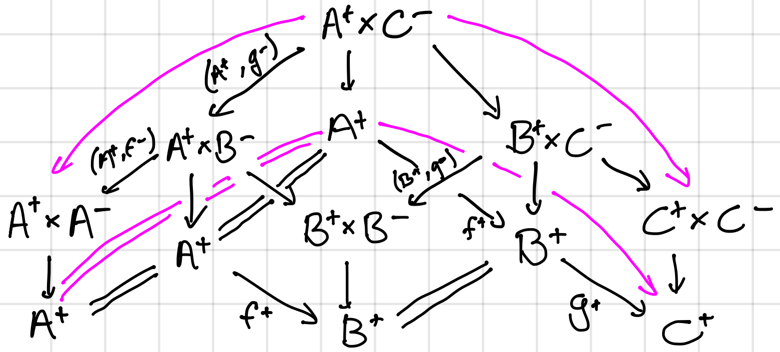
is a pullback

(b) Show that lens composition is given by the composition of their associated spans.

That is, given

$$\begin{pmatrix} f^- \\ f^+ \end{pmatrix} : \begin{pmatrix} A^- \\ A^+ \end{pmatrix} \leftrightarrows \begin{pmatrix} B^- \\ B^+ \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} g^- \\ g^+ \end{pmatrix} : \begin{pmatrix} B^- \\ B^+ \end{pmatrix} \leftrightarrows \begin{pmatrix} C^- \\ C^+ \end{pmatrix}$$

Show that the the pink span below corresponds



to the composite $\begin{pmatrix} g^- \\ g^+ \end{pmatrix} \circ \begin{pmatrix} f^- \\ f^+ \end{pmatrix}$, and that the top and bottom squares of the middle cube are pullbacks

⑧ A class of maps in a category is a class of maps $\mathcal{D} \subseteq \text{arrows}(\mathcal{C})$ such that if the following square commutes

$$\begin{tikzcd} D_0 \arrow[r, "i", "\sim"'] \arrow[d, "d_0"'] & D_2 \arrow[d, "d_1"] \\ D_1 \arrow[r, "j", "\sim"'] & D_3 \end{tikzcd}$$

and i and j are isomorphisms, then if $d_1 \in \mathcal{D}$, so is d_0.

A class of maps $\mathcal{D}$ is pullback stable if

For every $d : D_0 \to D_1$ in $\mathcal{D}$ and $f : C_1 \to D_1$ (arbitrary)
There is a pullback square

$$\begin{tikzcd} f^*D_0 \arrow[r, "\bar{f}"] \arrow[d, "f^*d"'] \arrow[dr, phantom, "\lrcorner", very near start] & D_0 \arrow[d] \\ C_1 \arrow[r, "f"'] & D_1 \end{tikzcd}$$

with $f^*d \in \mathcal{D}$.

ⓐ Show that in a cartesian category, the class of left product projection $\pi : A \times X \to A$ is pullback stable.
↳ that is, of maps isomorphic to left product projections

ⓑ If $\mathcal{D}$ is any pullback stable class of maps in $\mathcal{C}$,

Then we get an indexed category

$$\mathcal{D}_{/(-)} : \mathcal{C}^{op} \to \mathcal{C}at$$

Where $\mathcal{D}_{/C}$ is the full subcategory of the slice category $\mathcal{C}/C$ spanned by the maps in $\mathcal{D}$

That is, objects are $\begin{tikzcd} D_0 \arrow[d, "d_0"'] \\ C \end{tikzcd} \in \mathcal{D}$, and maps are $\begin{tikzcd} D_0 \arrow[rr, "f"] \arrow[dr, "d_0"'] & & D_1 \arrow[dl, "d_1"] \\ & C & \end{tikzcd}$

And $\mathcal{D}_{/f} : \mathcal{D}_C \to \mathcal{D}_{C'}$ (for $f : C' \to C$)

is given by pullback. Verify that this is an indexed cat.

(8) (c) Show that $\int^{c:\mathcal{C}} \mathcal{D}/c$ is the full subcat of the arrow cat $\mathcal{C}^{\downarrow}$ spanned by the maps in $\mathcal{D}$.

(d) Show that a map in $\int^{c:\mathcal{C}} \mathcal{D}/c$, considered as a square

$$\begin{tikzcd} D_0 \arrow[r, "\bar{f}"] \arrow[d, "d_0"'] & D_2 \arrow[d, "d_1"] \\ D_1 \arrow[r, "f"'] & D_3 \end{tikzcd}$$

is vertical iff f is iso and cartesian iff the square is a pullback.

(9) For $\mathcal{D} = \{\text{product projections}\}$ in a cartesian category $\mathcal{C}$, the indexed category

$$\mathcal{D}/_{(-)} : \mathcal{C}^{op} \longrightarrow Cat$$

is called the "simple fibration".

Show that $\mathcal{D}/_{(-)}$-lenses are simple lenses.

$$Lens_{\mathcal{D}/_{(-)}} \cong Lens_{\mathcal{C}}$$

(10) Show that in the Grothendieck construction of an indexed category that:

(a) For any cartesian $f : X \to Y$ and vertical $g : A \to Y$ There is a pullback

$$(*) \quad \begin{tikzcd} B \arrow[r, "\bar{f}"] \arrow[d, "\bar{g}"'] & A \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

where $\bar{f}$ is cartesian and $\bar{g}$ is vertical.

(b) Show that any square $(*)$ where $f, \bar{f}$ are cartesian and $g, \bar{g}$ are vertical is a pullback.

Extracises

(A) Prove that any very lawful lens in Set has "constant complement".

That is, if $\begin{pmatrix} f^+ \\ f^- \end{pmatrix} : \begin{pmatrix} A \\ A \end{pmatrix} \leftrightarrows \begin{pmatrix} B \\ B \end{pmatrix}$ satisfies

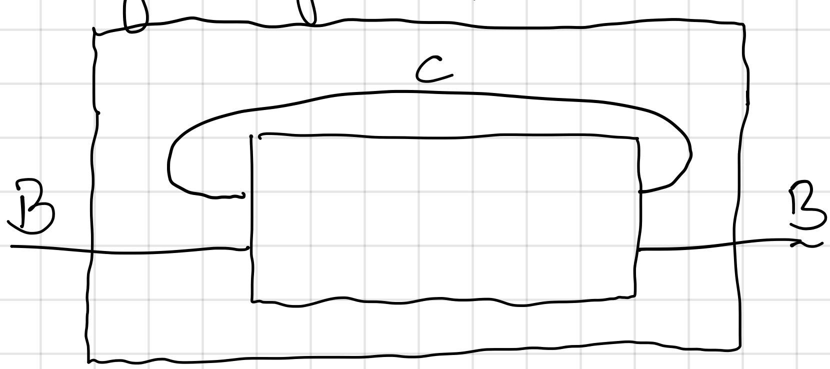
$$\begin{aligned}
&(\text{get put}) && f^+(f^-(a,b)) = b \\
&(\text{put get}) && f^-(a, f^+(a)) = a \\
&(\text{put put}) && f^-(f^-(a,b), b') = f^-(a,b')
\end{aligned}$$

Then there is a set C and a map $c : A \to C$ so that

(a) $(f^+, c) : A \to B \times C$ is an iso

(b) $f^-(a,b) = (f^+, c)^{-1}(b, c(a))$

Show that these lenses are the interpretations of the following wiring diagram



(B) Compute what lenses are for your favorite indexed category $\mathcal{F} : \mathcal{C}^{op} \to \mathrm{Cat}$.

(C) Dependent lenses are lenses for $\mathrm{Set}/_{(-)} : \mathrm{Set}^{op} \to \mathrm{Cat}$.

What should "dependent wiring diagrams" be?