# Markov Categories: a tutorial – exercise sheet

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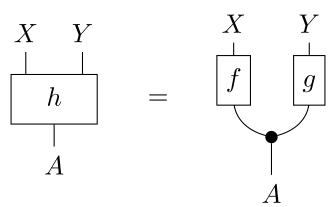
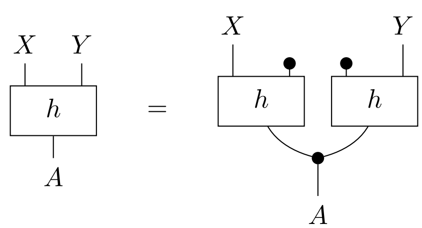
## 1 Conditional independence and determinism

**Exercise 1.1.** Show that if a joint morphism $h : A \to X \otimes Y$ decomposes as a product,




then it is the product of its marginals, i.e.




**Exercise 1.2.** Show that the deterministic morphisms of $\mathsf{FinStoch}$ are exactly the matrices with only entries 0 and 1.

For people who know some measure theory: what is the analogous statement in $\mathsf{Stoch}$?

**Exercise 1.3.** Prove that for a Markov category $\mathsf{C}$, the following conditions are equivalent.

(i) Every morphism is deterministic;

(ii) The copy maps are the components of a natural transformation (between which functors?);

(iii) $\mathsf{C}$ is cartesian monoidal.

Hint: show that if a morphism $h : A \to X \otimes Y$ is deterministic, then it is always making $X$ and $Y$ conditionally independent given $A$.

**Exercise 1.4** (bonus)**.** Show that the conditions of the previous exercise are also equivalent to the following.

(iv) Every joint morphism $h : A \to X \otimes Y$ makes $X$ and $Y$ conditionally independent given $A$.

## 2 Almost-sure equality and conditionals

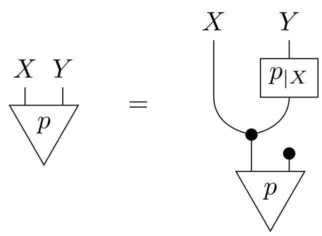
**Exercise 2.1.** Show that in FinStoch, given a state $p : I \to X$ and morphisms $f, g : X \to Y$, we have that $f = g$ $p$-almost surely if and only if the set

$$\{x \in X : f(x) \neq g(x)\}$$

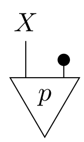
has $p$-measure zero.

If you have a background in measure theory, show that the same is true in BorelStoch.

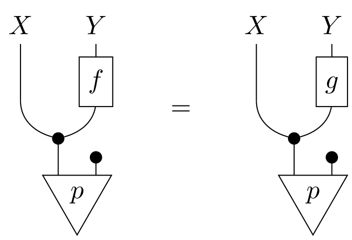
**Exercise 2.2.** Suppose that the following conditional distribution exists.



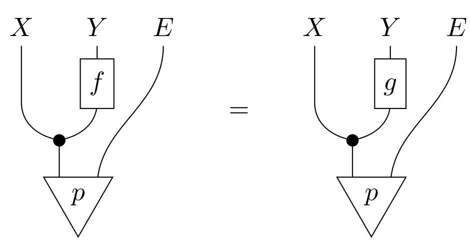

Show that any (other) morphism $f : X \to Y$ is a conditional of $p$ given $X$ if and only if it is $p_X$-almost surely equal to $p_{|X}$, where $p_X$ denotes the following marginal.




**Exercise 2.3.** Suppose that a Markov category C has conditional distributions. Prove the **equality strengthening** property, namely that if




then we also have the following stronger condition.




**Exercise 2.4.** Use the previous exercise to show that composition in the category ProbStoch(C) is well-defined.

**Exercise 2.5** (bonus)**.** For people who know dagger categories: show that if $\mathsf{C}$ has conditional distributions, $\mathsf{ProbStoch}(\mathsf{C})$ is a monoidal dagger category.

**Exercise 2.6** (bonus)**.** For people who know lenses: construct a "forgetful" retrofunctor (a.k.a. cofunctor) $\mathsf{ProbStoch}(\mathsf{C}) \to \mathsf{C}$.

# 3 Monads and Kolmogorov products

**Exercise 3.1.** Let $(\mathsf{D}, \times, 1)$ be a *cartesian* monoidal category. Let $(P, \mu, \eta, \nabla)$ be a monad on $\mathsf{D}$ which is

- **Affine**, i.e. $P1 \cong 1$,
- **Monoidal**, or **commutative**, with structure maps

$$PA \times PB \xrightarrow{\nabla} P(A \times B)$$

  satisfying the usual associativity and unitality conditions.

Show that the Kleisli category of $P$ is canonically a Markov category, with the copy and discard maps induced by those of $\mathsf{C}$.

**Exercise 3.2.** For people who know measure theory: show that $\mathsf{Stoch}$ is the Kleisli category of the Giry monad on the category $\mathsf{Meas}$ of measurable spaces and measurable maps.

Hint: why is a Markov kernel a Kleisli morphism?

**Exercise 3.3.** Let $\mathsf{C}$ be a Markov category constructed as in Exercise 3.1. Prove that sampling from a product distribution yields the same result as sampling from the factors independently:

$$\begin{array}{ccc} PX \otimes PY & \xrightarrow{\nabla} & P(X \otimes Y) \\ & \searrow^{\mathsf{samp}\otimes\mathsf{samp}} & \downarrow{\scriptstyle \mathsf{samp}} \\ & & X \otimes Y \end{array}$$

**Exercise 3.4.** Prove that a Kolmogorov product $\bigotimes_{i\in I} X_i$ in a Markov category $\mathsf{C}$ reduces to the cartesian product $\prod_{i\in I} X_i$ if we restrict to the category $\mathsf{C}_{\mathrm{det}}$ of deterministic morphisms.

**Question 3.5** (research-level)**.** For people who know measure theory: a measure is called **s-finite** (not to be confused with $\sigma$-finite) if it can be written as a countable sum of finite measures. Do s-finite measures form a monad, analogous to the Giry monad? If so, does the monad restrict to the category of standard Borel spaces?

## References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